Feeling number: grounding number sense in a sense of quantity

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Abstract Drawing on results from psychology and from cultural and linguistic studies, we argue for an increased focus on developing quantity sense in school mathematics. We explore the notion of "feeling number," a phrase that we offer in a two-fold sense—resisting tendencies to feel numb-er (more numb) by developing a feeling for numbers and the quantities they represent. First, we distinguish between quantity sense and the relatively vague notion of number sense. Second, we consider the human capacity for quantity sense and place that in the context of related cultural issues, including verbal and symbolic representations of number. Third, and more pragmatically, we offer teaching strategies that seem helpful in the development of quantity sense coupled with number sense. Finally, we argue that there is a moral imperative to connect number sense with such a quantity sense that allows students to feel the weight of numbers. It is important that learners develop a feeling for number, which includes a sense of what numbers are and what they can do.

Keywords: quantity sense, number sense, critical mathematics education, numerosity, number systems

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> Upon investigation of the natural numerical sense of the average twentieth century person, one discovers that lurking behind the inherited scheme for writing large numbers is a very limited feeling for what the number symbols are conveying about quantity. (Barrow, 1992, p. 41)

Number permeates the existences of citizens of the modern, Western world. As Fey (1990) has pointed out, numeration systems have become indispensable. They are necessary for making sense of the world. Lives have become, quite literally, numbered.

Few phenomena have resisted quantification—and this truth is powerfully demonstrated in history books, current media reports, and efforts to predict what is on the horizon. Past, present, and future are framed by and infused with numbers. In the first two pages of today's newspaper, for example, there are reports on the popularity of local politicians, gambling debts, prison terms, expected lifespan of diabetics, miscalculations in retirements savings, and economic projections. In fact, every single item in those two pages—including the paid advertisements and indices—foregrounds numbers.

Our aim is not to critique societal tendencies to quantify, rank-order, and assign values, although we do feel that would be a worthy project. Rather, our intention is to explore the notion of "feeling number," a phrase that we offer in a two-fold sense—feeling numb-er (more numb) and developing a feeling for numbers. Developing a sense of feel for number can, we suggest, mitigate some of the numbness that accompanies quantification.

First, we distinguish between quantity sense and the relatively vague notion of number sense. Second, we consider the human capacity for quantity sense and place it in the context of related cultural issues. Third, we offer some teaching strategies that seem helpful in coupling the development of quantity sense with number sense. Finally, we address a moral question: should students' number sense be grounded in quantity sense?

1 Quantity Sense versus Number Sense

The term "number sense" has been a prominent part of the mathematics education vocabulary for several decades now. In 1987, the Commission on Standards for School Mathematics, initiated by the National Council of Teachers of Mathematics (NCTM), offered the following definition:

> Children with good number sense (1) have well-understood number meanings, (2) have developed multiple relationships among numbers, (3) recognize the relative magnitudes of numbers, and (4) know the relative effect of operating on numbers. (p. 37)

This definition, with its battery of skills and understandings is in line with scholarly accounts of research on children's number sense (e.g. Jordan et al., 2007).

In these definitions there is recognition of the need for a sense of quantity and for ability with computations, but these are not explicitly connected. In school curricula based on NCTM recommendations, we note attention to a sense of quantity (or magnitude) of numbers in the early years and a gradual foregrounding of computation until it eclipses experiences with quantity in the middle years. As number is slowly but surely abstracted from context, students become numb to the meaning of the numeric symbols they learn to manipulate.

Taking the complete set of NCTM standards as elaboration of the definition given above, we find at least nominal valuing of contextualization in number work at all levels. Thus the NCTM calls for more than automaticity, which is described by psychologists as "the phenomenon that a skill can be performed with minimal awareness of its use (Axtell et al., 2008, p. 527). However, we argue that typical applications of mathematics, which appear mostly in word problems, may not connect sufficiently to students' personal contexts. Gerofsky (2004) showed that word problems, which haven't changed much in 4000 years, are actually set up to encourage students to ignore contexts. For example, in a typical word problem involving someone named John measuring rice, "John", "rice" and even the numbers are placeholders that could be substituted for other things or values—the aspect that cannot be ignored are the cues that indicate apparently necessary operations. With the prevalence of such relatively meaningless word problems, it would appear that number sense is taken to be the ability to work with numbers in the absence of context and intention.

With curricula explicitly or implicitly based on NCTM standards, the objectives of developing number sense and quantity sense are conflated under the rubric of "number sense" and, within this conflation, calculations are strongly privileged. It is beyond the scope of our argument to demonstrate this privileging. Rather we take it as a given in our argument to attend more carefully to quantity sense.

Quantity sense is very different from, but not unrelated to computation-based number sense or what NCTM documents often refer to as 'operation sense.' The following chart summarizes and contrasts some of the key differences. We elaborate on some of these differences later in the article.

Quantity sense	**Computation-based number sense**
• a sense of *how much*, *size*; a *feel* for amounts and magnitudes	• an ability to manipulate numbers appropriately, especially when combining them
• innate for small quantities, elaborated through experience	• cultural, symbol-based, and entirely learned
• shared with many species	• uniquely human[1]

2 Quantity sense and its limitations

The significance of educating quantity sense is not a small point because, as Barrow (1992) highlighted, the human capacity to imagine quantity is very limited. This can be demonstrated through a simple experiment. Close your eyes and imagine quantities of dots, beginning with one and moving upward to ten. (This demonstration is more effective if you actually try this before moving to the next paragraph.)

If you were able to envision those quantities, chances are that you relied on some grouping or parsing strategies for the larger amounts. The tasks of imagining one or two items are simple for most. Picturing three items is a bit more difficult, and most project a triangular arrangement in order to make the task easier. With prompting, the triangle can be rearranged into three items in a row. A similar thing usually happens with four. People tend to imagine items organized into a square, but with a little effort can line them up.

The limits on this capacity usually start to appear with quantities greater than four. Five is readily imagined if one pictures a pentagon or the familiar dice arrangement, but holding an image of five objects in a row is quite a challenge. It becomes even more challenging for six.

[1] There is some evidence that certain species—including primates, dolphins, and, in particular, parrots—can make effective and abstract use of numbers. However, to date, all evidence is based on capacities to master particular aspects of human numeration systems. See Scholtyssek (2006).

Imagining two rows of three or a hexagon is fairly easy. Most humans cannot imagine six items in a row without parsing them into subgroups—a detail that was first discussed by Miller (1956) in his reports on the limitations of consciousness.

The point being made is that humans have very limited capacities to imagine quantity—which neurological study indicates to be rooted in an innate capacity that is elaborated through experience. Dehaene (1997) summarized some of the research on humans' inborn capacity to distinguish small quantities at a glance. Called *subitizing*, from the Latin for "sudden," the evidence suggests that humans are able to distinguish between very small quantities within a few hours of birth.

This competence is not based on number. Rather, it is more a matter of the *feel* for magnitude, and it is a capacity that humans share with many species. Barrow (1992) explained,

> Experiments reveal that some animals possess a rudimentary *notion of quantity* although not the *notion of counting*. … [We] know that … cats and dogs are able to detect a small change in quantity. Experiments with birds reveal that in general they can choose the larger of two piles of seeds if the variation in quantity is 3 and 1, 4 and 2, or 4 and 3, but generally lose track when the choice is between piles of 4 and 5. (p. 168, emphasis added)

Barrow's distinction between the *notion of quantity* and the *notion of counting* (i.e., the use of numbers) is an important one. With numbers, as with other knowledge technologies, humans are able to amplify biological predispositions. This distinction is especially significant when considering research on what psychologists call *numerosity*. This research, which is "one of the oldest domains of experimental psychology" (Luwel et al., 2005, p. 449), considers children's ability to ascertain the number of objects in a group (i.e. the group's numerosity). In these tests, which focus on numbers less than 100, children use multiple strategies in the tests the psychologists use—strategies that often involve a mix of counting and sensing quantity without counting (e.g., Sophian and Chu, 2008).

We feel that undue attention to numerosity directs too much attention to abstract understandings of number. As recognized by Rousselle and Noel (2008), "Numerosity is one of the most abstract dimensions of our environment" (p. 544) because it is not a physical property, but rather a property of a grouping, which is an action or a way of seeing. With this abstraction, it is all too easy to relate children's abilities to achievement in mathematics classes (e.g. Luwel et al., 2005), which tends to be evaluated on the basis of skills with computations, instead of in relation to their application of these abilities outside of the classroom, which is often asserted to be the reason for developing numerosity judgment (e.g. Luwel et al., 2005).

Our worry as educators is that number sense is often divorced from quantity sense, and that curricular and pedagogical emphases in school mathematics actually contribute to their separation. Before exploring the implications of this situation, we do acknowledge the importance of the conceptual leap associated with the separation of number from quantity. We agree with Barrow (1992) in his assertion that "the greatest step of all" in the emergence of modern mathematics was "the recognition of an abstract concept of number divorced from collections of particular objects" (p. 41), but we add that this 'great step' is both a step toward something and a step away from something else. As with most 'great steps,' the one described by Barrow is a cultural step.

3 Cultural aspects of quantity sense

Numeration is at the heart of this abstraction. As Donald (2001) described the situation,

> Whereas earlier humans had to depend entirely on their biology—that is, on their brains—to remember, modern humans can employ a huge number of powerful external symbolic devices to store and retrieve cultural knowledge. (p. 262)

Donald's point is powerfully demonstrated in the contrast between those cultures that have developed sophisticated numeration systems and those that have not. Cultural distinctions that relate to the extent and nature of linguistic representation of number have been interrogated to test the Whorfian hypothesis (see, e.g. Whorf, 1956), which, in its strongest form, states that a person's or culture's language repertoire determines the nature and content of thought. (In its weaker form, Whorf's hypothesis states that language *influences* thought.) Studies of the Pirahã (Gordon, 2004) and Mundurukú (Pica et al., 2004) Amazonian Indians of Brazil have either refuted the strong form of Whorf's hypothesis, at least in relation to quantity sense, or demonstrated that number sense can develop independently from quantity sense. The people in these cultures are able to distinguish between quantities though their number systems are limited in extent and precision. For example, the Pirahã people use three number words, which roughly represent one, two, and many. They do not linguistically represent precise counts beyond two, but they are reasonably fluent with comparisons of quantity.

Though cultures that rely on rudimentary number representations may not need words for large numbers, they do need to communicate large quantities and thus have ways of expressing them. This phenomenon is evident in ethnomathematical research amongst Mi'kmaw (an Aboriginal group) communities in Canada, from which we note that specific number words are relatively unnecessary for much everyday mathematics (Wagner and Lunney, 2006). For example, the actual number of potatoes someone collects for a meal is not as important as the quantity (or volume) because individual potatoes vary considerably in size, especially when they are not graded (for size and quality) as they are in most grocery stores. The number of potatoes is relatively abstract compared to the amount or volume of potatoes. Lunney Borden (2010) has noted that number appears more in Mi'kmaw games than in practical applications. In this culture and in others (e.g. Macpherson's (1987) description of an Inuit child's mathematics), quantity sense is more spatial than it is numeric.

Thus it is possible to have a kind of quantity sense with limited connection to number. However, we will claim that this kind of quantity sense can be and ought to be extended to connect with number. Quantity sense alone is not sufficient to interpret large-scale phenomena that become increasingly important with globalization. Something is lost in the decoupling of number and concrete experience, as there are conceptual and cultural implications of coupling and decoupling.

Significantly, because numerosity is the description of a process, not of a thing, structures of language and patterns of interaction can either draw attention to this *process* of coupling, or to the *result* of a coupling. To illustrate, number words in Mi'kmaq are verbs, not nouns or adjectives, as they are in English (Lunney Borden, 2010). Objects can be said to be two-ing (or coupling), but it cannot be said that they *are* two. For Mi'kmaw people, it would be more natural to think of the human act of numbering objects than it would be for unilingual English-speakers.

Another feature of Mi'kmaq that relates to abstraction is that number words are different for different kinds of objects. Swetz (2009) has noted that more descriptive and thus less abstract numerics abound in world languages, especially in more traditional cultures. Languages vary in the degree to which they draw attention to the objects being counted or to the abstract number. Together, these two structures of language—numbers that work as verbs or as adjectives (or verbs) that depend on the nouns to which they refer—make a link between number and objects more obvious. By contrast, English grammar, which favours the use of nouns to represent numerosity, facilitates the abstract manipulation of number independent from the concrete objects they represent. Again, this relates to Whorf's hypothesis. Not only the existence of number words, but also the grammatical structuring of the language used with numbers influence the way a culture experiences the world.[2]

Coupling and decoupling is cultural—in different cultural contexts different things are coupled regularly, and different contexts invite coupling or decoupling. Not only does the ability to distinguish quantities vary amongst individuals, but also amongst cultural groups, suggesting that experience plays a role in this ability. Zaslavsky (1999), who described cultures similar to the Pirahã and Mundurukú cultures, in that they have one-two-many forms of representation for numbers, explained that such systems are actually used to represent larger numbers with precision using an approach similar to exponentiation but more meaningfully related to their environment. Zaslavsky reported that even non-literate individuals from such cultures out-performed Yale undergraduates on tests that involve identifying quantities without counting. This suggests that quantity sense can be developed. This possibility is significant to our promotion of developing quantity sense.

4 Quantity sense and number systems

Number systems transcend or at least span cultures. These symbolic systems, like verbal language, can vary in the way they draw attention to or away from quantity. Numeration systems have tended to be structured around the natural limits of human ability to distinguish quantity. Close study of this competence reveals that counting becomes necessary when there are more than a few items involved. The evidence shows that, even though the recognitions might all feel instantaneous, it takes longer to identify larger quantities (Dehaene, 1997).

Exemplifying the recognition of human limitations to recognize quantity, the Roman and Mayan systems introduced new symbols whenever the recognized-with-confidence limit of four was surpassed. Such number systems were made useful in the fact that they could be used to compress information. Clearly, CCLXXIX is more concise and accessible than a pile of 279 pebbles or a set of 279 tally marks. Further to this, we note that one aspect of the current Base-10 numeration system seems to be deliberately structured around the limits of human ability to recognize quantity. The fact that spaces (or commas, periods, or hyphens) are used to separate digits into clusters of three makes it much easier to recognize and compare large numbers—even if it does little to connect these numbers to their associated quantities.

Unlike symbol-substitution systems, place value systems are not anchored in our abilities to subitize. For example, if you know that the Roman numeral 'VII' can be expanded into IIIIIII, you can probably work out the quantity associated with VIII and perhaps even XIII. 'X' always represents 10; 'M' always 1000. However, the same is not the case for the Arabic '7,' '8,' and '13.' For example, '7,' depending on its position, could represent seven, seven billion, seven tenths, one seventh and the list goes on. These Arabic symbols simply do not point to quantities in the same way as the Roman symbols.

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[2] We suggest that Whorf's hypothesis applied to number representation could inspire investigation of the relationships between cultures' forms of representing large numbers and their practices that relate to relatively large quantities. Such practices may include the accumulation of possessions, structures of governance, and the organization of trade.

The principal issue here is that a place value system is based on exponentiation (e.g. $111 = 10^2 + 10^1 + 10^0$). In contrast, tallying systems are based on counting and addition, symbol-substitution systems are based on addition and multiplication (e.g., $CCLXXXI = 2 \times C + 1 \times L + 3 \times X + 1 \times I$). The exponential basis of place value systems, we believe, prompts users to approach large quantities in *metaphoric*, not *literal* terms. That is, 3 000 000 is understood as three measures, not in terms of three millions. This raises the question whether the quantities that underlie such large numbers can be imagined.

5 Developing a feel for numbers

Some years ago, on a television news broadcast, an effort was made to illustrate the magnitude of the $50.56 billion 1-year loss of J.D.S. Uniphase. The reporter indicated that, if this sum were converted to a stack of paper currency, the pile would be about 2.3 times the height of the world's tallest free-standing structure at the time, Toronto's CN Tower (i.e., in total, about 1270 m). The punch line was that the stack that represents this towering loss would not consist of $1 bills, but of $10 000 bills.

As an attempt to promote an understanding of an immense quantity, our sense is that this one was not particularly effective. Too many of the parts fall outside the realm of the typical viewer's experience. Few, for example, will have a strong sense of the height of the CN-Tower, and perhaps even fewer have much of a familiarity with $10 000 bills—let alone tower-tall stacks of them. However, it is a different situation for the journalist who developed this illustration, and who used calculations to grasp the enormity of the loss with personally meaningful experiences. This journalist used both number and quantity sense together, unlike the viewer audience who need not to have done so.

It is not uncommon for journalists to try to help their audience appreciate the enormity of certain numbers that appear in their reports. These attempts, like the J.D.S. Uniphase report, demonstrate a recognition that quantity sense is lacking. Such efforts are bound to fail given our society's numbness to quantity, a numbness that is only hardened by the sheer quantity of large numbers with which people are bombarded daily and for which a small minority of people have relevant personal experience. How does one respond to the predictable failure of such attempts to educate society's quantity sense?

The problem of conceptualizing large numbers begs the question: Is it at all possible to develop a feel for such immense quantities, or are we condemned to dealing with only the numbers and a vague sense that they are larger than we can imagine? Though Zaslavsky's (1999) evidence that quantity sense varies between cultures suggests that quantity sense is learned, at least for relatively small numbers, to question the viability of educating quantity sense goes beyond the questions addressed by such studies, which have investigated peoples' fluency with relatively small numbers, like 2 and 80. We want to address very large quantities as well: is it possible for humans to develop a sense of huge quantities and the numbers that represent these quantities? Our sense, and our experience, is that it is indeed possible, but efforts must be anchored in individual students making comparisons that relate closely to their experiences.[3] Simply giving students an image to associate with a large quantity is not nearly as effective as

[3] In our increasingly multicultural societies, it is becoming more difficult to choose settings that are closely related to the experiences of *all* the students in any given classroom. Their experiences vary considerably. Thus, we encourage teachers to adapt our strategies or use different strategies with similar characteristics to attend to their particular students' experiences.

prompting them to draw on their experiences to develop their own images. To that end, we have developed and used three teaching strategies, each focused on different quantities. They are not particularly unique strategies. We present them here to point out features of classroom activity that can bring to attention the tension between number sense and quantity sense.

The first strategy is focused on amounts under 100. Very simply, it involves making estimations of quantities after very brief glimpses of, for example, the contents of a box of matches spread out onto the glass plate of an overhead projector. This activity is similar to tests psychology researchers administer to establish people's numerosity judgement. Our experience has been that students can quickly develop a sense of how many. Recently, for example, four Grade 4 classes (9- to 11-year-olds) participated in a four-week effort to improve quantity sense. At the start of each daily math class, they were given a one-second glimpse of a quantity of pill-sized dots projected onto a screen with an overhead projector. Estimates and strategies were discussed. At the beginning of the experience, estimates varied widely (e.g., between 25 and 100 for an image with 72 dots – noting that they were aware of the maximum of 100). About 10% of the students were consistently able to provide an estimate within 10% of the actual number. By the end of the month, more than 95% of the students were able to provide spontaneous estimates that were within 10% of the actual number and estimates were rarely more than 15% off. This activity develops quantity sense for relatively small numbers and tacitly engages computation-based number sense because students' strategies often draw upon calculations.

Our second teaching strategy is intended for amounts between 100 and 10 000 and is an elaboration of the first. Using pictures of trees, brick walls, crowds, flocks, or whatever students experience in groups of 100 to 10 000, we invite students to estimate the number of leaves, bricks, people, birds, etc.—first as a guesstimate and then through a more careful analysis. This activity engages quantity recognition with images from students' experiences. It actually takes up surprisingly little time, as the two-dimensionality of a photograph lends itself to quick area-based calculations (e.g., if there are 9 heads in a typical 1 cm^2 area of a photograph of a crowd, the total number of heads is readily estimated). Once again, with practice, students can become quite adept at generating reasonable estimates—at least reasonable in the sense that estimates are consistently an appropriate level of magnitude. Again, quantity sense is the focus, and number sense is engaged as students' chose strategies (such as proportional reasoning) that require calculation. They choose from their individual repertoires, which expand to include the strategies described by their peers. This activity extends students' sense of quantity even further than the first activity.

Our third teaching strategy is less focused on the capacity to estimate quantities and more concerned with the development of one's appreciation for truly large amounts, namely those in excess of 10 000. It bears similarities to activities promoted by Ronau (1988) in an NCTM teacher's journal, and to Fermi problems. Our strategy revolves around counting rice. All that is required is a measuring tool (we use a small bowl to mete out approximately 150 mL portions), flat surfaces, and a bag of rice (we use regular, long grain). We also recommend sheets of ledger-sized dark paper, which can be very useful for containing efforts, using area-based strategies, and helping with the cleanup. The orienting question is a simple, "Approximately how many grains of rice are there in a small bowl?"

Having used this activity dozens of times in contexts that range from Grade 4 classrooms to teacher in-service sessions, we can report that there are many, many ways to tackle the problem successfully, including successive halving and redoubling, counting (e.g., 100), building matching piles and multiplying the number of piles by the approximate number in each, converging groups of piles to develop larger piles (e.g. 10 at 100 = 1000), measurement-based strategies using a convenient tool (e.g., pen-cap) coupled to a counting-type strategy, area-calculation strategies (e.g., spreading the rice out over an area; counting the grains in a sub-region; multiplying), volume-calculation strategies (e.g., forming a pyramid, calculating its volume, and dividing by the volume of a grain), and sampling (e.g., counting out and marking a pre-selected number of grains that are returned to the bowl; random samples are then drawn and an estimate of the total is calculated based on the ratio of marked:unmarked grains).

A cursory review of these tactics reveals that they touch on virtually every topic in a typical middle school curriculum—involving, for example, exponentiation, fraction work, area and volume calculations, and statistical sampling. More subtly, the range of strategies has also proven to be a rich site for discussing how and where errors are introduced. In general, for example, methods that involve more steps (i.e., ones that have more places to introduce and/or amplify errors) generate poorer estimates.

Typically, it takes about 10 to 20 minutes (depending mainly on the age of the participants) for groups to generate a confident response. For the most part, these responses hover around 10 000 (± 1 500). Once established, we use this as a basis to discuss larger amounts. Discussions are framed by the following sorts of questions and tasks:

- How large of a container would be needed to hold 1 000 000 grains of rice? (possible answer: 15 L, approx. one large sack)
- 1 000 000 000? (possible answer: approx. a medium-sized North American bedroom full)
- Illustrate the probability of winning the Lotto 6/49 jackpot. (i.e., of pre-selecting 6 randomly chosen numbers out of 49 possibilities; possible answer: approx. one grain out of a stack of 14 sacks)
- Illustrate how many people there are in the world. (possible answer: approx. six or seven bedrooms—or a small North American apartment-full)
- Illustrate the portion of the world's population that lives in Canada. (possible answer: 30 sacks in an apartment-full of rice)
- Illustrate Bill Gates' wealth relative to, say, an annual teacher's salary. (possible answer: a school full of rice *versus* a small bag)
- Consider and illustrate the "chessboard rice-doubling problem" (i.e., how many grains of rice will you have in total if, on a chessboard, you place 1 grain on the first square, 2 on the second, 4 on the third, and so on? This task can be subdivided into questions such as, at which point will you have a bowl-full? A sack-full? A room-full? etc.)

We pause here to re-emphasize that the value of the strategy described here is in the students' active roles in connecting experiences of quantity with their computation skills. It is an emphasis that might be illustrated by drawing a contrast between the activity and various visual images, now quite common in classrooms and texts, that represent a million – a million dots on a poster, a million pennies, a tree with a million leaves, a crowd of a million people, even a million grains of rice – all of which we regard as appropriate efforts to display quantity, but none of which attempt to connect such portrayals to students' arithmetic work with number.

This third activity is like the first two in that it engages both number and quantity senses, but in a way it is different because it begins with instructions to calculate and tacitly demands connections to personal experience. The order is reversed. The others *begin* with an experience, and expect students to introduce calculations. In this activity, students do calculations and seek to *connect* to their experiences, drawing on their repertoires of contexts to calculate reasonably-sized "containers" to represent the large numbers. Nevertheless, students are not told what computations to make, so they still need to draw on their individual and collective number sense repertoires.

The development of quantity sense is dependent on making connections between comparisons of large quantities and the critical concerns of students, some of which would align with social concerns. We argue that it is important for teachers to distinguish quantity sense from calculation skills to ensure that both are addressed and that students are engaged in challenges that draw on both at once, thus engaging the tension between them—the tension between abstraction and groundedness in order to make number work more meaningful.

6 Quantity sense, number sense, and critical sense

We must confess that our own pedagogical intentions here are not strictly with supporting learners' capacities to imagine and compare large quantities. It is also to enable them to interpret and respond critically to the barrage of numbers that they meet each and every day. Nevertheless, with this confession we recognize that our concern is shared by many educators. This goal appears in many curricula as rationale for attention to numbers. However, we want to be more specific with a call for more attention to quantity in work with numbers, instead of separating attention to quantity from computation work.

The NCTM's (2000) elaborated section on number in the high school standards has included a claim that quantity sense is needed by the citizenry—"As citizens, students will need to grasp the difference between $1 billion, the cost of a moderate-sized government project, and $1 trillion, a significant part of the national budget" (p. 290)—but the NCTM's examples of mathematics class activities involving numbers point to their use as abstractions. In the overview of number standards, the value of number sense seems to be for equipping students to take on further mathematical procedures, which tend to be even greater abstractions as the students 'progress' through the school ranks.

The numbers are manipulated as though they are things themselves. They are seen as nouns, not as adjectives that describe very real things. The NCTM claim that democracy requires number sense appears to justify the pursuit of abstraction. However, its argument supports the development of quantity sense more than it supports the development of number sense. We reiterate our concern to distinguish between these two senses.

Our promotion of quantity and number sense warrants consideration of the reasons for developing these senses in our children, and ultimately, in our society. It is not good enough to say we should develop quantity sense simply because it can be done. Perhaps an abstracted sense of number would serve society more than a number sense grounded in quantity sense. All number work involves some abstraction by the very nature of numbers as symbols, but what varies is the extent to which the initial problems and the results are grounded in meaningful contexts.

As with any curriculum choices, it is necessary to base choices on a social imperative for the skill or understanding in question. As shown by Zevenbergen (2004), people's perceptions of what number-related experiences should be included in curriculum vary considerably. She encourages the reconsideration of what seems basic because "in these New Times, new forms of knowledge and knowing are needed [and] old basics may not be relevant for these times" (p. 100).

We have referred to evidence that cultures vary in their people's quantity sense (though the research we referred to does not use the term 'quantity sense'). Historical investigation also reveals cultural development in the way the Western world has dealt with the tension between number sense and quantity sense. Concern for what is lost in the decoupling associated with number manipulation is not new. Championing the push for abstraction and ever greater number symbols is not new either; Swetz (2009) reported that some people have 'rated' cultures on the basis of the extent of their numbers, rating a culture with words and symbols for relatively large numbers as more advanced than one with less extensive symbols.

Tahta (1991) detailed the Medieval suspicion of numeric computation as it challenged the more familiar abacus, which gave a stronger sense of association between quantities and their representations. He also showed how simple substitution systems, like Roman numerals, are 'closer' to the quantities represented than place value systems. Perhaps it is significant that the Medieval tension between the algorists (people who preferred numeric algorithms and the associated abstraction of number) and the abacists (people who preferred the abacus and its close association between objects and number) was closely related to the development of capitalism as a powerful social force. This relationship, noted by Tahta, may relate to the current privileging of number sense over quantity sense.

Porter (1995), who is a historian, has investigated more recent history as he traced the Western world's pursuit of objectivity and its increasing trust in numbers. Like Tahta, he associated the development of human manipulation of numbers with sociocultural phenomena. Substantiating the NCTM's nod to citizenship, Porter described the development of quantification and explains reasons for its association with democracy. In addition to detailing historical examples from emerging democracies, in which attempts were made to educate society with quantified data, Porter described the continuing 'need' for quantified data in a democracy's bureaucracy:

> This is why a faith in objectivity tends to be associated with political democracy, or at least with systems in which bureaucratic actors are highly vulnerable to outsiders. […] The appeal of numbers is especially compelling to bureaucratic officials who lack the mandate of a popular election, or divine right. Arbitrariness and bias are the most usual grounds upon which such officials are criticized. (p. 8)

Porter continued with some strong words that alert us to the need for critical number sense grounded in quantity sense: "Quantification is a way of making decisions without seeming to decide" (p. 8). As politicians and bureaucrats use numbers to claim objectivity, to mask their biases and to legitimize their decisions, it could be said that the citizens, who have been enculturated in schools to put their trust in number, are being duped by number, not empowered to make informed decisions. And, of course, claims of objectivity are made by more people than politicians and bureaucrats. Children and adults need their number sense to be part of their *critical* sense.

Critical number sense requires a critical quantity sense that works with computational skills. If people believe that they can grasp large numbers, in the sense that they can hold them and manipulate them fluently, they may believe mistakenly that they have a grasp on these numbers' quantity. Thus, their 'informed' democratic decisions would rest on misunderstanding. To avert this problem, we argue that it is important to educate quantity sense with operation sense and to apply these senses to critical issues students face.

The readiness with which the popular press has picked up on the studies of Pirahã and Mundurukú numbers points to the currency of number sense in today's societies. We can imagine the many people who fear mathematics enjoying their read of the following excerpt from an article appearing in one of Canada's more respected newspapers: "1+1=2. Mathematics doesn't get any more basic than this, but even 1+1 would stump the brightest minds among the Pirahã tribe of the Amazon" (*Globe and Mail*, Friday, August 20, 2004). Though the authors of this article seem to have taken some liberty in their interpretation of Gordon's (2004) research, the currency of popular ridicule of poor number sense is instructive. While it seems somehow acceptable in our society to have a poor grasp of mathematics, it also seems acceptable to ridicule the naïveté of people who demonstrate poorer still number sense. This suggests either a tacit recognition of the value of both number and quantity sense, or a subtle self-indictment of our own naïveté, which may represent a (probably subconscious) awareness that we, as a society, are missing important understanding because we have insufficient sense of number and quantity.

7 Reflection

As we have drawn attention to the oft-forgotten connection between quantity and number, we have advocated mathematics classroom experiences that can help students feel the weight of number. It is our hope that such experiences will awaken them to be less numb to their physical and social environment and more aware of the role of mathematics in society. As with any consideration of what ought to be in children's curricula, this part of the discussion is fundamentally about values.

Alan Bishop, who is known for his attention to the cultural nature of mathematics education, noted that exponential-based numbers, which comprise a relatively abstract system of representing quantity, made colonialism possible (Bishop, 1990). We add that colonialist aspirations also made exponential-based numbers necessary, even before the relatively recent global colonisations: necessity is the mother of invention. This raises a question about the 'need' for a developed quantity sense, which could be seen as a technology, just as number sense can be seen as one. The relative absence of quantity sense can be read as confirmation that our present society's aspirations do not 'need' this sense. Furthermore, the absence could be read to mean that the society needs an *absence* of quantity sense. Indeed, how could the wealthy minority (including ourselves) go on living lives of excess with an awareness of the disparity in the world?

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